

Exactly solvable potentials with finitely many discrete eigenvalues of arbitrary choice

Ryu Sasaki

Department of Physics, Shinshu University,
Matsumoto 390-8621, Japan

Abstract

We address the problem of possible deformations of exactly solvable potentials having finitely many discrete eigenvalues of arbitrary choice. As Kay and Moses showed in 1956, reflectionless potentials in one dimensional quantum mechanics are exactly solvable. With an additional time dependence these potentials are identified as the soliton solutions of the KdV hierarchy. An N -soliton potential has the time t and $2N$ positive parameters, $k_1 < \dots < k_N$ and $\{c_j\}$, $j = 1, \dots, N$, corresponding to N discrete eigenvalues $\{-k_j^2\}$. The eigenfunctions are elementary functions expressed by the ratio of determinants. The Darboux-Crum-Krein-Adler transformation or the Abraham-Moses transformations based on eigenfunctions deletions produce lower soliton number potentials with modified parameters $\{c'_j\}$. We explore various identities satisfied by the eigenfunctions of the soliton potentials, which reflect the uniqueness theorem of Gel'fand-Levitan-Marchenko equations for separable (degenerate) kernels.

1 Introduction

In recent years, exactly solvable potentials in one dimensional quantum mechanics have aroused resurgent interest thanks to the discovery of certain new solvable potentials having the exceptional and the multi-indexed orthogonal polynomials as the main part of the eigenfunctions. These new solvable potentials are obtained by rational deformations of known solvable potentials; these are the radial oscillator, Pöschl-Teller and Coulomb potentials having infinitely many discrete eigenstates and Morse, Rosen-Morse, Eckart, hyperbolic Pöschl-Teller, $\text{sech}^2 x$ and the hyperbolic symmetric top potentials with finitely many discrete eigenstates. One conspicuous absence is the reflectionless potentials [1], or with the explicit time dependence the so-called N -soliton solutions [2] of the KdV hierarchy. For the reflectionless potentials derived by Kay and Moses [1] in 1956, all the eigenfunctions are exactly calculable for any number of arbitrarily given eigenvalues $\{-k_j^2\}$, $j = 1, \dots, N$. However, their exact solvability does not seem to be widely known among the present day researchers

of the subject. This is partly because the reflectionless potentials are *not shape invariant* and do not satisfy the well established sufficient condition of exact solvability.

In this paper we address the problem of possible solvable deformations of the reflectionless potentials in terms of the eigenfunctions, à la Darboux-Crum-Krein-Adler [3]-[5] and Abraham-Moses [6, 7]. They are known to generate reflectionless potentials [8]. Contrary to the naive expectation, these deformations of the general soliton solutions do not produce a new type of reflectionless potentials, or new species of soliton solutions. This is in good contrast to the known deformation examples like the multi-index [9, 10] and the exceptional orthogonal polynomials [11]–[18] cases. This fact is consistent with the uniqueness of the reflectionless potentials as the solutions of Gel'fand-Levitan-Marchenko equations [19] for separable (or degenerate) kernels. (See Appendix of Kay and Moses [1].) The non-deformation, in turn, could be understood as the consequences of many interesting identities satisfied by the reflections potentials and their eigenfunctions. We explore these identities as the characteristic properties of the reflectionless potentials. An attempt to deform $\text{sech}^2 x$ potentials with special t dependence to create integer speed solitons was reported recently [20].

The present paper is organised as follows. In section two, the explicit formulas of the reflectionless potential and the eigenfunctions are recapitulated for introducing necessary notation and for self-containedness. An alternative and intuitive derivation of the reflectionless potential and eigenfunctions is presented. The relationship between the reflectionless potentials and the soliton solutions of the KdV hierarchy is explained in some detail. Section three is the main body of the paper. Deformations of reflectionless potential by deleting single and multiple eigenstates via Darboux-Crum-Krein-Adler and Abraham-Moses transformations are performed explicitly. Several interesting Wronskian identities among the eigenfunctions of reflectionless (soliton) potentials are derived instead of new species of reflectionless potentials. The final section is for a summary and comments. The basic formulas of multiple Darboux and Abraham-Moses transformations are recapitulated in §A.1 and §A.2, respectively, for reference purposes.

2 Reflectionless potential and its eigenfunctions

Here we recapitulate the essence of the reflectionless potential and the corresponding eigenfunctions. Since most of the results are well-known for more than forty years, we will not give

the details of the derivation and refer to the original paper and related references [1, 2, 21]. Let us start with a reflectionless potential $U_N(x)$ in one dimensional quantum mechanics. It is defined on the entire real line $-\infty < x < \infty$ and it vanishes at $\pm\infty$. Its scattering wave solution is reflectionless:

$$\mathcal{H} = -\frac{d^2}{dx^2} + U_N(x), \quad (2.1)$$

$$\mathcal{H}\psi_k(x) = k^2\psi_k(x), \quad \psi_k(x) \sim \begin{cases} t(k)e^{ikx} & x \rightarrow +\infty \\ e^{ikx} & x \rightarrow -\infty \end{cases}, \quad k > 0. \quad (2.2)$$

It has N arbitrarily given discrete eigenvalues,

$$\mathcal{H}\phi_{N,j}(x) = \mathcal{E}_j\phi_{N,j}(x), \quad \mathcal{E}_j = -k_j^2, \quad j = 1, \dots, N, \quad (2.3)$$

with $0 < k_1 < k_2 < \dots < k_N$, corresponding to the poles of the transmission amplitude $t(k)$, on the positive imaginary k -axis, $k = ik_j$, $j = 1, \dots, N$. According to Kay and Moses [1], the potential is *everywhere negative*, $U_N(x) < 0$, and it has an expression

$$U_N(x) \stackrel{\text{def}}{=} -2\partial_x^2 \log u_N(x), \quad (2.4)$$

$$u_N(x) \stackrel{\text{def}}{=} \det A_N(x), \quad (A_N(x))_{mn} \stackrel{\text{def}}{=} \delta_{mn} + \frac{c_m e^{-(k_m+k_n)x}}{k_m + k_n}, \quad m, n = 1, \dots, N, \quad (2.5)$$

in which $\{c_m\}$ are arbitrary positive parameters. For real x and positive $\{k_j\}$ and $\{c_j\}$, the $N \times N$ matrix $A_N(x)$ is positive definite. The *logarithmic potential* $u_N(x)$ has a simple expansion [2]

$$u_N(x) = \sum_{\mu} \exp \left[\sum_{j=1}^N \mu_j \eta_j + \sum_{j<l} a_{jl} \mu_j \mu_l \right], \quad (2.6)$$

$$e^{\eta_j} \stackrel{\text{def}}{=} \frac{c_j}{2k_j} e^{-2k_j x}, \quad e^{a_{jl}} \stackrel{\text{def}}{=} \frac{(k_j - k_l)^2}{(k_j + k_l)^2}, \quad \mu_j = 0, 1, \quad (2.7)$$

in which \sum_{μ} means a summation over 2^N terms of $\mu_1 = 0, 1, \mu_2 = 0, 1, \dots, \mu_N = 0, 1$. This expression also shows that $u_N(x)$ is positive for real x and positive $\{k_j\}$ and $\{c_j\}$.

For *consecutive positive integers* $\{k_j\}$, and special values of the parameters $\{c_j\}$,

$$k_j = j, \quad c_j = \frac{(N+j)!}{j!(j-1)!(N-j)!}, \quad j = 1, \dots, N, \quad (2.8)$$

$$u_N(x) = e^{-N(N+1)x} (1 + e^{2x})^{N(N+1)/2}, \quad U_N(x) = -N(N+1) \text{sech}^2 x, \quad (2.9)$$

the general reflectionless potential $U_N(x)$ (2.4) reduces to the reflectionless $\text{sech}^2 x$ potential, which is known to be exactly solvable with the eigenvalues $\mathcal{E}_j = -j^2$, $j = 1, \dots, N$.

Except for possible complex zeros of $u_N(x)$, the potential $U_N(x)$ is *holomorphic*. At a complex simple zero x_0 of $u_N(x)$, $u_N(x) = (x - x_0)r_N(x)$, $r_N(x_0) \neq 0$, the reflectionless potential $U_N(x)$ (2.4) has a *regular singularity*

$$U_N(x) = \frac{2}{(x - x_0)^2} + O(x - x_0), \quad (2.10)$$

with the *characteristic exponents* 2 and -1 . This means that the solutions of the Schrödinger equations with the potential $U_N(x)$ are generically *monodromy free* on the complex x -plane.

The special form of the reflectionless potential (2.4) means that $U_N(x)$ can be derived by *multiple Darboux transformations* from the *trivial potential* $U \equiv 0$. The Schrödinger equation with $U \equiv 0$ has *square non-integrable solutions*

$$\psi_j(x) \stackrel{\text{def}}{=} e^{k_j x} + \tilde{c}_j e^{-k_j x}, \quad 0 < k_1 < k_2 < \cdots < k_N, \quad (-1)^{j-1} \tilde{c}_j > 0, \quad (2.11)$$

$$-\partial_x^2 \psi_j(x) = -k_j^2 \psi_j(x), \quad j = 1, \dots, N. \quad (2.12)$$

Their inverses $\{1/\psi_j(x)\}$ are locally square integrable at $x = \pm\infty$. With the above sign of the parameters $\{\tilde{c}_j\}$ (2.11) the Wronskian $(W[f_1, \dots, f_n](x) \stackrel{\text{def}}{=} \det(\partial_x^{j-1} f_k(x))_{1 \leq j, k \leq n})$ of these seed solutions $\{\psi_j\}$ is positive and it gives $u_N(x)$ upto a factor which is annihilated by ∂_x^2 after taking the logarithm:

$$W[\psi_1, \dots, \psi_N](x) = \prod_{j>l}^N (k_j - k_l) \cdot e^{\sum_{j=1}^N k_j x} u_N(x), \quad (2.13)$$

$$U_N(x) = -2\partial_x^2 \log W[\psi_1, \dots, \psi_N](x) = -2\partial_x^2 \log u_N(x). \quad (2.14)$$

Here we have redefined the coefficient of $e^{-2k_j x}$ in $u_N(x)$ to be $c_j/(2k_j)$, $c_j > 0$. Similar derivation of the reflectionless potential, without the eigenfunctions, was reported more than twenty years ago [22]. The general theory of Darboux transformation says that the above constructed $U_N(x)$ (2.14) has N -discrete eigenvalues $\mathcal{E}_j = -k_j^2$ with the eigenfunctions

$$\phi_{N,j}(x) \propto \frac{W[\psi_1, \dots, \check{\psi}_j, \dots, \psi_N](x)}{W[\psi_1, \dots, \psi_N](x)}, \quad j = 1, \dots, N, \quad (2.15)$$

in which $\check{\psi}_j$ means that $\psi_j(x)$ is excluded from the Wronskian. For derivation, see (A.8). By the same multiple Darboux transformation, the plane wave solution e^{ikx} ($k > 0$) of the $U \equiv 0$ Schrödinger equation is mapped to

$$e^{ikx} \rightarrow \frac{W[\psi_1, \dots, \psi_N, e^{ikx}](x)}{W[\psi_1, \dots, \psi_N](x)} \sim \begin{cases} \prod_{j=1}^N (ik - k_j) \cdot e^{ikx} & x \rightarrow +\infty \\ \prod_{j=1}^N (ik + k_j) \cdot e^{ikx} & x \rightarrow -\infty \end{cases}, \quad (2.16)$$

as the Wronskian of exponential functions is a van der Monde determinant:

$$W[e^{\alpha_1 x}, e^{\alpha_2 x}, \dots, e^{\alpha_M x}](x) = \prod_{1 \leq k < j \leq M} (\alpha_j - \alpha_k) \cdot e^{\sum_{j=1}^M \alpha_j x}.$$

This scattering wave solution has reflectionless asymptotic behaviour, which is consistent with (2.2). This is an alternative derivation of the reflection potential (2.4). *Its reflectionless property and the exact solvability are quite intuitively understood.*

Now we will comment on the relationship between the reflectionless potential (2.4),(2.5) and the N -soliton solution of the KdV hierarchy. By construction, the eigenvalues $\{-k_j^2\}$ are independent of the parameters $\{c_j\}$. Any continuous change of $\{c_j\}$ generate continuous *isospectral deformation* of the reflectionless potential $U_N(x)$. A special choice of t -dependence

$$c_j \rightarrow c_j e^{8k_j^3 t}, \quad j = 1, \dots, N, \quad (2.17)$$

changes the reflectionless potential $U_N(x)$ to an N -soliton solution $U_N(x; t)$ of the KdV equation [2, 21]:

$$U_N(x; t) \stackrel{\text{def}}{=} -2\partial_x^2 \log u_N(x; t), \quad (2.18)$$

$$u_N(x; t) \stackrel{\text{def}}{=} \det A_N(x; t), \quad (A_N(x; t))_{mn} \stackrel{\text{def}}{=} \delta_{mn} + \frac{c_m e^{-(k_m + k_n)x + 8k_m^3 t}}{k_m + k_n}, \quad (2.19)$$

$$0 = \partial_t U_N - 6U_N \partial_x U_N + \partial_x^3 U_N. \quad (2.20)$$

From now on we will abuse the language and call both $U_N(x)$ (2.4), (2.5) and $U_N(x; t)$ (2.18), (2.19) N -soliton solutions. In the KdV equation (2.20), the x and t dependence of $U_N(x; t)$ is suppressed for simplicity of presentation. More general time dependence

$$c_j \rightarrow c_j \exp\left[\sum_{n=1}^{\infty} (2k_j)^{2n+1} t_{2n+1}\right], \quad j = 1, \dots, N, \quad (2.21)$$

generates the N -soliton solution of the KdV hierarchy. Here t_{2n+1} ($t_3 \equiv t$) is the time parameter corresponding to the n -th involutive Hamiltonian of the KdV hierarchy. As the solution of the non-linear KdV equation (2.20), the overall normalisation of $U_N(x)$ including the sign is immaterial, since it can be absorbed by the rescaling of the coefficient of the nonlinear term. It should be stressed, however, that the overall scale with the sign -2 is essential for the potential of the Schrödinger equation as shown above.

The eigenfunctions $\{\phi_{N,j}(x)\}$ of the reflectionless potential $U_N(x)$ (2.4), (2.5) have a simple expression as the ratio of determinants:

$$\phi_{N,j}(x) \stackrel{\text{def}}{=} \frac{\tilde{u}_{N,j}(x)}{u_N(x)} e^{-k_j x}, \quad \tilde{u}_{N,j}(x) \stackrel{\text{def}}{=} \det \tilde{A}_{N,j}(x), \quad j = 1, \dots, N, \quad (2.22)$$

$$(\tilde{A}_{N,j}(x))_{mn} \stackrel{\text{def}}{=} \delta_{mn} + \frac{k_j - k_m}{k_j + k_m} \frac{c_m e^{-(k_m + k_n)x}}{k_m + k_n}, \quad m, n = 1, \dots, N. \quad (2.23)$$

In other words, $\tilde{u}_{N,j}(x)$ is obtained from $u_N(x)$ by the replacement

$$\tilde{u}_{N,j}(x) : c_m \rightarrow c_m \times \frac{k_j - k_m}{k_j + k_m}, \quad m = 1, \dots, N. \quad (2.24)$$

It is easy to see that the eigenfunction $\phi_{N,j}(x)$ is square integrable with the proper asymptotic behaviour:

$$\phi_{N,j}(x) \sim \begin{cases} e^{-k_j x} & x \rightarrow +\infty \\ \text{constant} \times e^{+k_j x} & x \rightarrow -\infty \end{cases}. \quad (2.25)$$

The *groundstate* eigenfunction $\phi_{N,N}(x)$ corresponding to the lowest eigenvalue $-k_N^2$ is *positive* $\phi_{N,N}(x) > 0$ since the matrix $\tilde{A}_{N,N}(x)$ is positive definite. It needs no explanation that the eigenfunctions for the time-dependent potential $U(x; t)$ (2.18), *i.e.* the N -soliton solution, are obtained from (2.22), (2.23) by the same replacement (2.17) or (2.21).

3 Deformations and identities

Here we will discuss possible deformations of the reflectionless potentials and soliton solutions, the main theme of the present paper. By using the eigenfunctions, one can construct an $(N - M)$ -soliton solution from an N -soliton solution. One naively expects that the resulting $(N - M)$ -soliton would retain the dependence on the original $2N$ parameters, creating new species of solitons. However, several attempts in terms of Darboux and Abraham-Moses transformations have failed to produce such new types of solitons. Non-existence of new species of solitons is consistent with the uniqueness theorem of reflectionless potentials (see, the Appendix of Kay-Moses original paper [1]). On the other hand, it means various identities satisfied by the soliton solutions, which do not seem to be widely recognised or discussed. The eigenfunctions of exactly solvable quantum mechanical systems are known to satisfy various interesting identities. See [23] for the Wronskian identities satisfied by the Hermite, Laguerre and Jacobi polynomials. Similar Casoratian identities satisfied by the classical orthogonal polynomials obeying second order difference equations, those for the Wilson and Askey-Wilson polynomials, are reported in [24].

Let us start with the standard Darboux-Crum [4] transformation by using the ground state eigenfunction as the seed solution (see (A.4)):

$$U_N(x) \rightarrow U_N^{(1)}(x) = U_N(x) - 2\partial_x^2 \log \phi_{N,N}(x) = -2\partial_x^2 \log \tilde{u}_{N,N}(x). \quad (3.1)$$

The resulting reflectionless potential is an $N - 1$ soliton solution depending on $2(N - 1)$ parameters $\{k_m, c_m^{(1)}\}$, $m = 1, \dots, N - 1$ obtained from those of the original one by replacement

$$c_m \rightarrow c_m^{(1)} \stackrel{\text{def}}{=} c_m \times \frac{k_N - k_m}{k_N + k_m}, \quad m = 1, \dots, N - 1. \quad (3.2)$$

This replacement rule could be interpreted as generalized shape invariance.

Next we deform the N soliton solution by using M distinct eigenfunctions specified by $\mathcal{D} \stackrel{\text{def}}{=} \{d_1, \dots, d_M\} \subset \{1, \dots, N\}$, that is by using

$$\{\phi_{N,d_1}(x), \dots, \phi_{N,d_M}(x)\}. \quad (3.3)$$

In other words, the solitons having the parameters $\{k_{d_1}, \dots, k_{d_M}\}$ are deleted from the original N -soliton solution with the parameter set $\{k_1, \dots, k_N\}$:

$$\{k_{d_1}, \dots, k_{d_M}\} \subset \{k_1, \dots, k_N\}.$$

This type of deformation is called Krein-Adler transformation [5]. The result is the $N - M$ soliton solution with the set of parameters $\{k_1, \dots, k_N\} \setminus \{k_{d_1}, \dots, k_{d_M}\}$ and $\{c_m^{(M)}\}$:

$$U_N^{(M)}(x) = -2\partial_x^2 \log \tilde{u}_{N,\mathcal{D}}(x), \quad (3.4)$$

$$\tilde{u}_{N,\mathcal{D}}(x) : c_m^{(M)} \stackrel{\text{def}}{=} c_m \times \prod_{j=1}^M \frac{k_{d_j} - k_m}{k_{d_j} + k_m}, \quad (3.5)$$

in which $\tilde{u}_{N,\mathcal{D}}(x)$ is obtained from $u_N(x)$ (2.5) by replacing c_m with the above $c_m^{(M)}$ (3.5).

This means the following Wronskian identity:

$$W[\phi_{N,d_1}, \dots, \phi_{N,d_M}](x) \propto \frac{\tilde{u}_{N,\mathcal{D}}(x) e^{-\sum_{j=1}^M k_{d_j} x}}{u_N(x)}, \quad (3.6)$$

as (A.9) says

$$U_N^{(M)}(x) = U_N(x) - 2\partial_x^2 \log W[\phi_{N,d_1}, \dots, \phi_{N,d_M}](x) = -2\partial_x^2 \log \tilde{u}_{N,\mathcal{D}}(x).$$

The positivity of $\tilde{u}_{N,\mathcal{D}}(x)$ is guaranteed if \mathcal{D} is chosen to satisfy the conditions [5]:

$$\prod_{j=1}^M (d_j - m) \geq 0, \quad m = 1, \dots, N. \quad (3.7)$$

With these conditions it is trivial to see the non-negativeness of $c_m^{(M)} \geq 0$ (3.5) for positive $c_m > 0$. These conditions are easily satisfied if \mathcal{D} consists of one or many pairs of two consecutive integers.

Next we create an $N - M$ -soliton solution from the N -soliton solution (2.4),(2.5) by using eigenstate deleting Abraham-Moses transformations (A.16)–(A.19). In this case the parameters $\{e_j\}$ are not arbitrary but are the norm of the seed functions $e_j = (\varphi_j, \varphi_j)$, which are the eigenfunctions. Here we use the standard notation for the inner product, $(f, g) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x)g(x)dx$. This is an artificial constraint to make the formulas (A.16)–(A.19) look symmetrical for addition and deletion. For $j \neq l$, $(\varphi_j, \varphi_l) = 0$ because of the orthogonality of the eigenfunctions and we obtain a positive definite expression of \mathcal{F}_M (A.16):

$$(\mathcal{F}_M)_{jl} = e_j \delta_{jl} - \langle \varphi_j, \varphi_l \rangle = \left(\int_{-\infty}^{\infty} - \int_{-\infty}^x \right) \varphi_j(y) \varphi_l(y) dy = \int_x^{\infty} \varphi_j(y) \varphi_l(y) dy. \quad (3.8)$$

(See (A.11) for the definition of $\langle f, g \rangle$.) In this form the deformed potential $U^{(M)}(x)$ (A.16),(A.19) is independent of the normalisation of the eigenfunctions. Let us use M distinct eigenfunctions (2.22) specified by $\mathcal{D} \stackrel{\text{def}}{=} \{d_1, \dots, d_M\} \subset \{1, \dots, N\}$, that is by using $\{\phi_{N,d_1}(x), \dots, \phi_{N,d_M}(x)\}$ (3.3). It is interesting to see that the integrand of $(\mathcal{F}_M)_{jl}$ (3.8) is a derivative of a function of the eigenfunction type as

$$\phi_{N,j}^2(x) = -\partial_x \left(\frac{\tilde{w}_{N,j}(x)}{u_N(x)} \cdot \frac{e^{-2k_j x}}{2k_j} \right), \quad (3.9)$$

$$\phi_{N,j}(x) \phi_{N,l}(x) = -\partial_x \left(\frac{\tilde{v}_{N;j,l}(x)}{u_N(x)} \cdot \frac{e^{-(k_j+k_l)x}}{k_j+k_l} \right), \quad \tilde{v}_{N;j,j}(x) \equiv \tilde{w}_{N,j}(x), \quad (3.10)$$

in which $\tilde{w}_{N,j}(x)$ and $\tilde{v}_{N;j,l}(x)$ are obtained from $u_N(x)$ by the following replacements of c_m :

$$\tilde{w}_{N,j}(x) : c_m \rightarrow c_m \times \frac{(k_j - k_m)^2}{(k_j + k_m)^2}, \quad \tilde{v}_{N;j,l}(x) : c_m \rightarrow c_m \times \frac{k_j - k_m}{k_j + k_m} \cdot \frac{k_l - k_m}{k_l + k_m}. \quad (3.11)$$

The above factor in $\tilde{w}_{N,j}$ is the *scattering phase shift* of the j -th and m -th solitons and it is the square of the factor appearing in $\tilde{u}_{N,j}$ of the eigenfunction (2.22). The $N - 1$ -soliton solution obtained by an eigenfunction deleting Abraham-Moses transformation (A.13) using $\phi_{N,j}(x)$ is

$$U_N(x) \rightarrow U_N^{(1)}(x) = U_N(x) - 2\partial_x^2 \log \int_x^{\infty} \phi_{N,j}^2(y) dy = -2\partial_x^2 \log \tilde{w}_{N,j}(x). \quad (3.12)$$

By repeating this process M times in terms of the eigenfunctions specified by \mathcal{D} (3.3) is

$$U_N(x) \rightarrow U_N^{(M)}(x) = -2\partial_x^2 \log \tilde{w}_{N,\mathcal{D}}(x), \quad (3.13)$$

in which $\tilde{w}_{N,\mathcal{D}}(x)$ is obtained from $u_N(x)$ by the replacement

$$\tilde{w}_{N,\mathcal{D}}(x) : c_m \rightarrow c_m \times \prod_{j=1}^M \frac{(k_{d_j} - k_m)^2}{(k_{d_j} + k_m)^2}. \quad (3.14)$$

This in turn means a determinant identity

$$\det \left(\int_x^\infty \phi_{N,d_j}(y) \phi_{N,d_l}(y) dy \right)_{1 \leq j, l \leq M} = \det \left(\frac{\tilde{v}_{N;d_j,d_l}(x)}{u_N(x)} \cdot \frac{e^{-(k_{d_j} + k_{d_l})x}}{k_{d_j} + k_{d_l}} \right)_{1 \leq j, l \leq M} \quad (3.15)$$

$$\propto \frac{\tilde{w}_{N,\mathcal{D}}(x)}{u_N(x)} e^{-2 \sum_{j=1}^M k_{d_j} x}. \quad (3.16)$$

We now turn to the clarification of the role played by the eigenstate adding Abraham-Moses transformations on the N -soliton solution (2.4),(2.5). As remarked in §A.2, these transformations are exactly iso-spectral when the seed solutions are the eigenfunction themselves. In order to fix the interpretation of the parameters $\{e_j\}$ in (A.13)–(A.19), let us use the *normalised* seed solutions $\hat{\phi}_j$, $(\hat{\phi}_j, \hat{\phi}_j) = 1$. For the N -soliton solution, they are

$$\hat{\phi}_{N,j}(x) \stackrel{\text{def}}{=} \sqrt{c_j} \phi_{N,j}(x), \quad (\hat{\phi}_{N,j}, \hat{\phi}_{N,j}) = 1, \quad j = 1, \dots, N, \quad (3.17)$$

for the relation (3.9) gives a simple way to normalise the eigenfunctions:

$$\frac{\tilde{w}_{N,j}(x)}{u_N(x)} \cdot \frac{e^{-2k_j x}}{2k_j} \rightarrow \begin{cases} 0 & x \rightarrow +\infty \\ \frac{1}{c_j} & x \rightarrow -\infty \end{cases}, \quad (\phi_{N,j}, \phi_{N,j}) = \frac{1}{c_j}. \quad (3.18)$$

The one eigenstate addition by using $\hat{\phi}_{N,j}(x)$ goes as follows (A.13):

$$\begin{aligned} e_j + \langle \hat{\phi}_{N,j}, \hat{\phi}_{N,j} \rangle &= e_j + 1 - \frac{\tilde{w}_{N,j}(x)}{u_N(x)} \cdot \frac{c_j e^{-2k_j x}}{2k_j} \\ &= (e_j + 1) \frac{\left\{ n_{N,j}(x) + \frac{e_j}{e_j + 1} \frac{c_j}{2k_j} e^{-2k_j x} \tilde{w}_{N,j}(x) \right\}}{u_N(x)} \\ &= (e_j + 1) \frac{\tilde{z}_{N,j}(x)}{u_N(x)}, \end{aligned} \quad (3.19)$$

in which $\tilde{z}_{N,j}(x)$ is obtained from $u_N(x)$ by the replacement:

$$\tilde{z}_{N,j}(x) : c_j \rightarrow \frac{e_j}{e_j + 1} c_j. \quad (3.20)$$

In short, the eigenstate adding Abraham-Moses transformation in terms of the j -th eigenfunction $\hat{\phi}_{N,j}(x)$ and e_j does not introduce a new independent parameter. It simply rescales

the corresponding j -th parameter c_j to $\frac{e_j}{e_j+1}c_j$, namely $u_N(x)$ to $\tilde{z}_{N,j}(x)$. Here we have used a simple identity of the logarithmic potential $u_N(x)$:

$$u_N(x) = u_{N,j}(x) + \frac{c_j}{2k_j}e^{-2k_jx}\tilde{w}_{N,j}(x), \quad u_{N,j}(x) \stackrel{\text{def}}{=} u_N(x)|_{c_j \rightarrow 0}, \quad (3.21)$$

which is obvious from the expansion formula (2.6)–(2.7).

By repeating this process M times in terms of the eigenfunctions specified by \mathcal{D} (3.3), we obtain

$$U_N(x) \rightarrow U_N^{(M)}(x) = -2\partial_x^2 \log \tilde{z}_{N,\mathcal{D}}(x), \quad (3.22)$$

in which $\tilde{z}_{N,\mathcal{D}}(x)$ is obtained from $u_N(x)$ by the replacement

$$\tilde{z}_{N,\mathcal{D}} : c_{d_j} \rightarrow \frac{e_{d_j}}{e_{d_j}+1}c_{d_j}, \quad j = 1, \dots, M, \quad c_l \rightarrow c_l, \quad l \notin \mathcal{D}. \quad (3.23)$$

This in turn means a determinant identity

$$\begin{aligned} & \det \left(e_{d_j} \delta_{jl} + \langle \hat{\phi}_{N,d_j}, \hat{\phi}_{N,d_l} \rangle \right)_{1 \leq j, l \leq M} \\ &= \det \left((e_{d_j} + 1) \delta_{jl} - \frac{\tilde{v}_{N;d_j,d_l}(x)}{u_N(x)} \cdot \frac{\sqrt{c_{d_j}c_{d_l}}e^{-(k_{d_j}+k_{d_l})x}}{k_{d_j} + k_{d_l}} \right)_{1 \leq j, l \leq M} \end{aligned} \quad (3.24)$$

$$\propto \frac{\tilde{z}_{N,\mathcal{D}}(x)}{u_N(x)}. \quad (3.25)$$

This type of iso-spectral transformations were reviewed in §7 of [25], with the integer solitons of the sech^2x potential, special cases of the general soliton solutions.

As is known [26, 27] the one eigenstate (φ) adding/deleting Abraham-Moses transformation is obtained by the ordinary Darboux transformation in terms of φ followed by another in terms of $\bar{\varphi}^{(1)} \stackrel{\text{def}}{=} \varphi^{-1}(e \pm \langle \varphi, \varphi \rangle)$, which is a particular solution of the first deformed Hamiltonian with $U^{(1)} = U - 2\partial_x \log |\varphi|$. Thus it is also called a *binary Darboux transformation* in some research group.

4 Summary and comments

Contrary to the naive expectation, the deformation of the N -soliton solution in terms of M distinct eigenfunctions specified by $\mathcal{D} = \{d_1, \dots, d_M\}$ does not produce new species of $(N-M)$ -soliton solutions depending on $2N$ independent parameters. The obtained $(N-M)$ -soliton solution depends on $2(N-M)$ independent parameters:

$$\{k_1, \dots, k_N\} \setminus \{k_{d_1}, \dots, k_{d_M}\}, \quad c_m^{(M)} = c_m \times \prod_{j=1}^M \left(\frac{k_{d_j} - k_m}{k_{d_j} + k_m} \right)^\Xi, \quad m \in \{1, \dots, N\} \setminus \mathcal{D}, \quad (4.1)$$

in which $\Xi = 1$ for the Krein-Adler (multiple Darboux) transformation (3.4)–(3.5) and $\Xi = 2$ for the multiple eigenstate deleting Abraham-Moses transformation (3.13)–(3.14). The failure to generate new species of soliton solutions, in turn, implies various Wronskian (determinant) identities (3.6), (3.15)–(3.16), (3.24)–(3.25) in a similar way as other exactly solvable quantum mechanical systems [23, 24]. We did not address the problem of solution generating transformations of non-linear PDE's, *e.g.* Bäcklund transformations.

After the discovery of solitons of KdV equation [28, 29], more general scheme of inverse scattering theory were developed by AKNS-ZS [30, 31], which covered the modified KdV, sine-Gordon and non-linear Schrödinger equations among others. It would be interesting to pursue similar goals with these soliton solutions [32]; if they are exactly solvable, if their deformations generate new types of soliton solutions or result in various identities, the relationship with infinitely many conserved quantities and the corresponding Hamiltonian structures, geometrical interpretations, etc [33, 34].

Another challenge is the discretized solitons, which pose almost the same questions as above from the point of view of discrete quantum mechanics [35]. The analogues of Darboux-Crum-Krein-Adler transformations in discrete quantum mechanics are known [36] and the analogues of the Wronskian identities, Casoratian identities, for exactly solvable systems are also reported [24].

Most of the shape invariant and exactly solvable quantum mechanical systems can also be solved exactly in the *Heisenberg picture* [37]. It is interesting to try and find the Heisenberg operator solutions for the reflectionless potentials.

Acknowledgements

R.S. thanks Jen-Chi Lee and Choon-Lin Ho for useful discussion and for the hospitality at National Chiao-Tung University, National Center for Theoretical Sciences (North) and National Taiwan University. He also thanks K. Takasaki, S. Tsujimoto and E. Date for useful discussion. R.S. is supported in part by Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology (MEXT), No.22540186.

Appendix

A Deformation schemes

Here we provide a brief summary of Darboux transformations and other methods of deformation of the potentials and solutions of Schrödinger equations;

$$\mathcal{H} = -\frac{d^2}{dx^2} + U(x), \quad \mathcal{H}\psi(x) = \mathcal{E}\psi(x) \quad (\mathcal{E}, U(x) \in \mathbb{C}), \quad (\text{A.1})$$

$$\mathcal{H}\varphi_j(x) = \tilde{\mathcal{E}}_j\varphi_j(x) \quad (\tilde{\mathcal{E}}_j \in \mathbb{C} ; j = 1, 2, \dots, M), \quad (\text{A.2})$$

The functions $\{\varphi_j(x), \tilde{\mathcal{E}}_j\}$ ($j = 1, 2, \dots, M$) are called *seed* solutions. The subsequent two subsections are for self-containedness.

A.1 Multiple Darboux transformation

By picking up one of the above seed solutions, say $\varphi_1(x)$, we form new functions with the above solution $\psi(x)$ and the rest of $\{\varphi_l(x), \tilde{\mathcal{E}}_l\}$ ($l \neq 1$):

$$\psi^{(1)}(x) \stackrel{\text{def}}{=} \frac{W[\varphi_1, \psi](x)}{\varphi_1(x)} = \frac{\varphi_1(x)\partial_x\psi(x) - \partial_x\varphi_1(x)\psi(x)}{\varphi_1(x)}, \quad \varphi_{1,l}^{(1)}(x) \stackrel{\text{def}}{=} \frac{W[\varphi_1, \varphi_l](x)}{\varphi_1(x)}. \quad (\text{A.3})$$

It is elementary to show that $\psi^{(1)}(x)$, $\varphi_1^{-1}(x) \stackrel{\text{def}}{=} \varphi_1(x)^{-1}$ and $\varphi_{1,l}^{(1)}(x)$ are solutions of a new Schrödinger equation of a deformed Hamiltonian $\mathcal{H}^{(1)}$

$$\mathcal{H}^{(1)} = -\frac{d^2}{dx^2} + U^{(1)}(x), \quad U^{(1)}(x) \stackrel{\text{def}}{=} U(x) - 2\partial_x^2 \log|\varphi_1(x)|, \quad (\text{A.4})$$

with the same energies \mathcal{E} , $\tilde{\mathcal{E}}_1$ and $\tilde{\mathcal{E}}_l$:

$$\mathcal{H}^{(1)}\psi^{(1)}(x) = \mathcal{E}\psi^{(1)}(x), \quad \mathcal{H}^{(1)}\varphi_1^{-1}(x) = \tilde{\mathcal{E}}_1\varphi_1^{-1}(x), \quad (\text{A.5})$$

$$\mathcal{H}^{(1)}\varphi_{1,l}^{(1)}(x) = \tilde{\mathcal{E}}_l\varphi_{1,l}^{(1)}(x) \quad (l \neq 1). \quad (\text{A.6})$$

By repeating the above Darboux transformations M -times, we obtain new functions

$$\psi^{(M)}(x) \stackrel{\text{def}}{=} \frac{W[\varphi_1, \varphi_2, \dots, \varphi_M, \psi](x)}{W[\varphi_1, \varphi_2, \dots, \varphi_M](x)}, \quad (\text{A.7})$$

$$\check{\varphi}_j^{(M)}(x) \stackrel{\text{def}}{=} \frac{W[\varphi_1, \varphi_2, \dots, \check{\varphi}_j, \dots, \varphi_M](x)}{W[\varphi_1, \varphi_2, \dots, \varphi_M](x)} \quad (j = 1, 2, \dots, M), \quad (\text{A.8})$$

which satisfy an M -th deformed Schrödinger equation with the energies \mathcal{E} and $\tilde{\mathcal{E}}_j$ [3]:

$$\mathcal{H}^{(M)} = -\frac{d^2}{dx^2} + U^{(M)}(x), \quad U^{(M)}(x) \stackrel{\text{def}}{=} U(x) - 2\partial_x^2 \log|W[\varphi_1, \varphi_2, \dots, \varphi_M](x)|, \quad (\text{A.9})$$

$$\mathcal{H}^{(M)}\psi^{(M)}(x) = \mathcal{E}\psi^{(M)}(x), \quad \mathcal{H}^{(M)}\check{\varphi}_j^{(M)}(x) = \tilde{\mathcal{E}}_j\check{\varphi}_j^{(M)}(x) \quad (j = 1, 2, \dots, M). \quad (\text{A.10})$$

These multiple Darboux transformations are *essentially iso-spectral*, upto a finite number of added or deleted energy levels depending on the properties of the seed functions $\{\varphi_j(x), \tilde{\mathcal{E}}_j\}$ ($j = 1, 2, \dots, M$). In order to avoid singularities of the new potential $U^{(M)}(x)$, the Wronskian of the seed functions $W[\varphi_1, \varphi_2, \dots, \varphi_M](x)$ should not vanish on the real x -axis.

A.2 Multiple Abraham-Moses transformation

Next let us introduce the Abraham-Moses transformations [6, 7]. For simplicity of the presentation, we will restrict ourselves to utilise the solutions and seed solutions which are locally square integrable at $x = -\infty$. For a pair of real functions f and g , let us introduce a new function $\langle f, g \rangle$ by integration:

$$\langle f, g \rangle(x) \stackrel{\text{def}}{=} \int_{-\infty}^x dy f(y)g(y) = \langle g, f \rangle(x), \quad (\text{A.11})$$

$$\langle f, g \rangle(-\infty) = 0, \quad \langle f, g \rangle(+\infty) = (f, g) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x)g(x)dx. \quad (\text{A.12})$$

Note that $\frac{d}{dx}\langle f, g \rangle(x) = f(x)g(x)$. Throughout this paper, we use the simplified notation $\langle f, g \rangle \equiv \langle f, g \rangle(x)$, so long as no confusion arises.

For a seed solution, say φ_1 , with the energy $\tilde{\mathcal{E}}_1$, an Abraham-Moses transformation for adding/deleting one bound state with the energy $\tilde{\mathcal{E}}_1$, is defined as follows:

$$U(x) \rightarrow U^{(1)}(x) \stackrel{\text{def}}{=} U(x) - 2\partial_x^2 \log(e_1 \pm \langle \varphi_1, \varphi_1 \rangle), \quad e_1 > 0, \quad (\text{A.13})$$

$$\varphi_1 \rightarrow \varphi_1^{(1)} \stackrel{\text{def}}{=} \frac{\varphi_1}{e_1 \pm \langle \varphi_1, \varphi_1 \rangle}, \quad \mathcal{H}^{(1)}\varphi_1^{(1)} = \tilde{\mathcal{E}}_1\varphi_1^{(1)}, \quad (\text{A.14})$$

$$\psi \rightarrow \psi^{(1)} \stackrel{\text{def}}{=} \psi \mp \varphi_1^{(1)}\langle \varphi_1, \psi \rangle, \quad \mathcal{H}^{(1)}\psi^{(1)} = \mathcal{E}\psi^{(1)}. \quad (\text{A.15})$$

For the eigenstate addition, we choose the upper sign and $e_1 > 0$ is arbitrary. In this case a non-square integrable seed solution $(\varphi_1, \varphi_1) = \infty$ is mapped to an eigenstate $\varphi_1^{(1)}$ with the energy $\tilde{\mathcal{E}}_1$. If φ_1 is an eigenstate, $\varphi_1^{(1)}$ is also an eigenstate with the energy $\tilde{\mathcal{E}}_1$ and its normalisation is changed. In this case, the transformation is exactly iso-spectral and no eigenstate is added. For the eigenstate deletion, we choose the lower sign and e_1 is the norm of the eigenstate φ_1 , $e_1 \stackrel{\text{def}}{=} (\varphi_1, \varphi_1)$. The transformed state $\varphi_1^{(1)}$ is no longer square integrable, $(\varphi_1^{(1)}, \varphi_1^{(1)}) = \infty$, *i.e.* the eigenstate is deleted.

By repeating these Abraham-Moses transformations, we arrive at multiple eigenstate adding/deleting transformations [7, 38]:

$$U^{(M)}(x) = U(x) - 2\partial_x^2 \log \det(\mathcal{F}_M), \quad (\text{A.16})$$

$$\psi^{(M)} = \psi \mp \sum_{j,l=1}^M \varphi_j (\mathcal{F}_M^{-1})_{jl} \langle \varphi_l, \psi \rangle, \quad \mathcal{H}^{(M)} \psi^{(M)} = \mathcal{E} \psi^{(M)}, \quad (\text{A.17})$$

$$\varphi_j^{(M)} = \sum_{l=1}^M (\mathcal{F}_M^{-1})_{jl} \varphi_l, \quad \mathcal{H}^{(M)} \varphi_j^{(M)} = \tilde{\mathcal{E}}_j \varphi_j^{(M)}, \quad (j, l = 1, \dots, M), \quad (\text{A.18})$$

in which \mathcal{F}_M is an $M \times M$ symmetric and positive definite matrix depending on the seed solutions $\{\varphi_j\}$ ($j = 1, \dots, M$) defined by:

$$(\mathcal{F}_M)_{jl} \stackrel{\text{def}}{=} e_j \delta_{jl} \pm \langle \varphi_j, \varphi_l \rangle, \quad e_j \begin{cases} > 0 \text{ arbitrary} & \text{addition} \\ \stackrel{\text{def}}{=} (\varphi_j, \varphi_j) & \text{deletion} \end{cases}, \quad (j, l = 1, \dots, M). \quad (\text{A.19})$$

The positive definiteness of \mathcal{F}_M guarantees the regularity of the deformed potential.

References

- [1] I. Kay and H. M. Moses, “Reflectionless transmission through dielectrics and scattering potentials,” J. Appl. Phys. **27** (1956) 1503-1508.
- [2] R. Hirota, “Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons,” Phys. Rev. Lett. **27** (1971) 1192-1194.
- [3] G. Darboux, *Théorie générale des surfaces* vol 2 (1888) Gauthier-Villars, Paris.
- [4] M. M. Crum, “Associated Sturm-Liouville systems,” Quart. J. Math. Oxford Ser. (2) **6** (1955) 121-127, [arXiv:physics/9908019](#).
- [5] M. G. Krein, “On continuous analogue of a formula of Christoffel from the theory of orthogonal polynomials,” (Russian) Doklady Acad. Nauk. CCCP **113** (1957) 970-973; V. É. Adler, “A modification of Crum’s method,” Theor. Math. Phys. **101** (1994) 1381-1386.
- [6] P. B. Abraham and H. E. Moses, “Changes in potentials due to changes in the point spectrum: Anharmonic oscillators with exact solutions,” Phys. Rev. **A22** (1980) 1333-1340.
- [7] S. Odake and R. Sasaki, “Non-polynomial extensions of solvable potentials à la Abraham-Moses,” J. Math. Phys. **54** (2013) 102106 (19pp), [arXiv:1307.0931\[math-ph\]](#).

- [8] C.-L. Ho, J.-C. Lee and R. Sasaki, “Scattering amplitudes for multi-indexed extensions of solvable potentials,” *Annals of Physics*, (2014) in press, [arXiv:1309.5471 \[quant-ph\]](#).
- [9] S. Odake and R. Sasaki, “Exactly solvable quantum mechanics and infinite families of multi-indexed orthogonal polynomials,” *Phys. Lett.* **B702** (2011) 164-170, [arXiv:1105.0508 \[math-ph\]](#).
- [10] D. Gómez-Ullate, N. Kamran and R. Milson, “Two-step Darboux transformations and exceptional Laguerre polynomials,” *J. Math. Anal. Appl.* **387** (2012) 410-418, [arXiv:1103.5724 \[math-ph\]](#).
- [11] D. Gómez-Ullate, N. Kamran and R. Milson, “An extension of Bochner’s problem: exceptional invariant subspaces,” *J. Approx Theory* **162** (2010) 987-1006, [arXiv:0805.3376 \[math-ph\]](#); “An extended class of orthogonal polynomials defined by a Sturm-Liouville problem,” *J. Math. Anal. Appl.* **359** (2009) 352-367, [arXiv:0807.3939 \[math-ph\]](#).
- [12] C. Quesne, “Exceptional orthogonal polynomials, exactly solvable potentials and supersymmetry,” *J. Phys.* **A41** (2008) 392001 (6pp), [arXiv:0807.4087 \[quant-ph\]](#).
- [13] B. Bagchi, C. Quesne and R. Roychoudhury, “Isospectrality of conventional and new extended potentials, second-order supersymmetry and role of PT symmetry,” *Pramana J. Phys.* **73** (2009) 337-347, [arXiv:0812.1488 \[quant-ph\]](#).
- [14] S. Odake and R. Sasaki, “Infinitely many shape invariant potentials and new orthogonal polynomials,” *Phys. Lett.* **B679** (2009) 414-417, [arXiv:0906.0142 \[math-ph\]](#).
- [15] C. Quesne, “Solvable rational potentials and exceptional orthogonal polynomials in supersymmetric quantum mechanics,” *SIGMA* **5** (2009) 084 (24pp), [arXiv:0906.2331 \[math-ph\]](#).
- [16] S. Odake and R. Sasaki, “Another set of infinitely many exceptional (X_ℓ) Laguerre polynomials,” *Phys. Lett.* **B684** (2010) 173-176, [arXiv:0911.3442 \[math-ph\]](#).
- [17] C.-L. Ho, S. Odake and R. Sasaki, “Properties of the exceptional (X_ℓ) Laguerre and Jacobi polynomials,” *SIGMA* **7** (2011) 107 (24pp), [arXiv:0912.5447 \[math-ph\]](#).

- [18] S. Odake and R. Sasaki, “Extensions of solvable potentials with finitely many discrete eigenstates,” J. Phys. **A46** (2013) 235205 (15pp), [arXiv:1301.3980 \[math-ph\]](#).
- [19] I. M. Gel’fand and B. M. Levitan, “On the determination of a differential equation from its spectral function,” (Russian) Izvestiya Akad. Nauk SSSR. Ser. Mat. **15** (1951) 309-360 (Amer. Math. Soc. Transl. Ser.2 **1** (1955) 253-304); V. A. Marchenko, “Spectral theory of Sturm-Liouville operators,” Kiev (1972) (Russian); K. Chadan and P. C. Sabatier, *Inverse problems in quantum scattering theory*, second edition, Springer Verlag, New York (1988).
- [20] C.-L. Ho and J.-C. Lee, “Multi-indexed extensions of soliton potential and extended integer solitons of KdV equation,” [arXiv:1401.1150 \[quant-ph\]](#).
- [21] S. Tanaka and E. Date, *KdV equation*, Kinokuniya, Tokyo (1979) (Japanese).
- [22] V. B. Matveev and M. A. Salle, *Darboux transformations and solitons*, Springer-Verlag, Berlin Heidelberg (1991).
- [23] S. Odake and R. Sasaki, “Krein-Adler transformations for shape-invariant potentials and pseudo virtual states,” J. Phys. **A46** (2013) 245201 (24pp), [arXiv:1212.6595 \[math-ph\]](#).
- [24] S. Odake and R. Sasaki, “Casoratian identities for the Wilson and Askey-Wilson polynomials,” [arXiv:1308.4240 \[math-ph\]](#).
- [25] F. Cooper, A. Khare and U. Sukhatme, “Supersymmetry and quantum mechanics,” Phys. Rep. **251** (1995) 267-385.
- [26] W. A. Schnitzer and H. Leeb, “Generalized Darboux transformations: classification of inverse scattering methods for the radial Schrödinger equation,” J. Phys. **A27** (1994) 2605-2614.
- [27] B. F. Samsonov, “On the equivalence of the integral and the differential exact solution generation methods for the one-dimensional Schrödinger equation,” J. Phys. **A28** (1995) 6989-6998.
- [28] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, “Method for solving the Korteweg-de Vries equation,” Phys. Rev. Lett. **19** (1967) 1095-1097.

- [29] P.D. Lax, “Integrals of nonlinear equations of evolution and solitary waves,” *Comm. Pure Appl. Math.* **21** (1968) 467-490.
- [30] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, “Nonlinear-evolution equations of physical significance,” *Phys. Rev. Lett.* **31** (1973) 125-127.
- [31] V.E. Zakharov and A.B. Shabat, “Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media,” *Soviet Phys. JETP* **34** (1972) 62-69.
- [32] R. Hirota, “Exact solution of the modified Korteweg-de Vries equation for multiple collisions of solitons,” *J. Phys. Soc. Japan* **33** (1972) 1456-1458; “Exact solution of the sine-Gordon equation for multiple collisions of solitons,” *J. Phys. Soc. Japan* **33** (1972) 1459-1463.
- [33] M. Wadati, H. Sanuki and K. Konno, “Relationship among inverse method, Bäcklund transformation and an infinite number of conservation laws,” *Prog. Theor. Phys.* **53** (1974) 419-436.
- [34] R. Sasaki, “Soliton equations and pseudospherical surfaces,” *Nucl. Phys.* **B154** (1979) 343-357.
- [35] S. Odake and R. Sasaki, “Discrete quantum mechanics,” (Topical Review) *J. Phys.* **A44** (2011) 353001 (47pp), [arXiv:1104.0473\[math-ph\]](#).
- [36] L. García-Gutiérrez, S. Odake and R. Sasaki, “Modification of Crum’s theorem for ‘discrete’ quantum mechanics,” *Prog. Theor. Phys.* **124** (2010) 1-24, [arXiv:1004.0289\[math-ph\]](#).
- [37] S. Odake and R. Sasaki, “Unified theory of annihilation-creation operators for solvable (‘discrete’) quantum mechanics,” *J. Math. Phys.* **47** (2006) 102102 (33pp), [arXiv:quant-ph/0605215](#); “Exact solution in the Heisenberg picture and annihilation-creation operators,” *Phys. Lett.* **B641** (2006) 112-117, [arXiv:quant-ph/0605221](#).
- [38] L. Trlifaj, “The Darboux and Abraham-Moses transformations of the one-dimensional periodic Schrödinger equation and inverse problems,” *Inverse Problems* **5** (1989) 1145-1155.